ON THE CALCULATION OF WAVE FIELDS IN ELASTIC HOMOGENEOUS MEDIA WITH PLANE PARALLEL PARTITIONING BOUNDARIES

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At the present time, quantitative studies of exact solutions of dynamic problems in the theory of elasticity are usually conducted by means of approximate asymptotic methods. These methods vary depending on the portion of the disturbed layered medium being investigated. In a number of cases it is difficult to estimate the accuracy and region of applicability of the formulas that are obtained.

For a more accurate and complete investigation which determines the entire wave field in a uniform manner, contour integrals are sometimes used (uniformity is very important here in order to have a simple standard scheme for numerical computations). The description of the solution is thereby reduced to integrals distributed along segments of the real axis which are evaluated by usual methods of numerical integration. As a point of departure, one can take any of the known forms of exact solutions [1-4] for concentrated excitations. The transition to distributed excitations, as is well known, can then be effected by application of the principle of superposition. It is convenient to use, for example, solutions in the form of formulas which are given in [3]. The present paper is addressed to the task of the reduction of these formulas to the simplest real integrals suitable for computation.

1. According to [3] the exact solution for a concentrated excitation of the Heaviside ε function type is expressible (for an isolated wave) in cylindrical coordinates r, θ, z as a double integral of the form

$$u_\nu = \int_0^\infty \left\{ \int_{\sigma - i\infty}^{\sigma + i\infty} G_\nu(\zeta)\, e^{k\varphi(\zeta)}\, d\zeta \right\} J_\nu(kr)\, dk \qquad (\nu = 0, 1; \sigma > 0) \qquad (1.1)$$

or its derivatives with respect to the time t. In this formula, $G_\nu(\zeta)$ denotes an algebraic expression which depends on the shear modulus and the propagation velocity of the wave, and $\varphi(\zeta)$ is a linear function depending on the time t and thicknesses of the layers h_j (including the moving coordinate z)

$$\varphi(\zeta) = bt\zeta - \sum_{j=0}^{n} q_j h_j \sqrt{1 + \gamma_j^2\zeta^2} \qquad (\arg\sqrt{1+\gamma_j^2\zeta^2} = 0 \quad \text{for } \zeta > 0) \tag{1.2}$$

Here, as the arbitrary parameter b, one usually chooses the smallest (of those possible) of the velocities of propagation of transverse waves in the layered medium, γ_j is the ratio of b to the velocity of propagation of a wave of some type in the jth layer, and q_j is the number of passages in the jth layer by the wave of velocity $b\gamma_j^{-1}$.

Let H be some linear parameter, say, the thickness of one of the layers; we introduce nondimensional quantities

$$\tau = bt/H, \qquad x = r/H \tag{1.3}$$

Then Formulas (1.1) and (1.2) take on the forms

$$u_\nu = \frac{1}{H}\int_0^\infty \left\{ \int_{\sigma - i\infty}^{\sigma + i\infty} G_\nu(\zeta)\, e^{k\varphi(\zeta)}\, d\zeta \right\} J_\nu(kx)\, dk \qquad (\nu = 0, 1;\ \sigma > 0) \tag{1.4}$$

$$\varphi(\zeta) = \tau\zeta - \sum_{j=0}^{n} q_j \frac{h_j}{H} \sqrt{1 + \gamma_j^2\zeta^2} \tag{1.5}$$

The integration with respect to k can be carried out if the contour in Equation (1.4) $\sigma - i\infty$, $\sigma + i\infty$ is suitably deformed (as was indicated in [5]). As a result, the wave field can be represented by single-fold integrals

$$u_0 = \frac{1}{H}\int_l \frac{G_0(\zeta)\, d\zeta}{\sqrt{[\varphi(\zeta)]^2 + x^2}}, \qquad u_1 = \frac{1}{H}\,\frac{1}{x}\int_l G_1(\zeta)\left(1 + \frac{\varphi(\zeta)}{\sqrt{[\varphi(\zeta)]^2 + x^2}}\right) d\zeta \tag{1.6}$$

For l it is here necessary to take the contour which previously enveloped the singular points, $\sigma - i\infty$, $\sigma + i\infty$ but with $\sigma = 0$ or with a sufficiently small $\sigma > 0$. The branch of the radical [illegible] is fixed by the condition [6] that the radical should equal [illegible] as a consequence of this $\arg\sqrt{[\varphi(\zeta)]^2 + x^2} = 0$ [illegible]

In order to uniformize all of the radicals we make cuts from the branch points which extend to infinity parallel to the imaginary axis but not intersecting the real axis.

If the wave field in back of a dilatational wave front is examined it is found that there are two complex conjugate branch points of the radical $\sqrt{(|\varphi(\zeta)|^2 + x^2)}$ to the right of the contour l. If one transfers the point of observation from in back of a dilatational wave front to the region ahead of the front, then the aforementioned branch points remaining to the right of the contour l collapse onto the imaginary axis (as a limiting transition from a distributed to a concentrated excitation indicates [5]).

All remaining branch points and possible poles in the integrand are located to the left of the contour l and turn out always to be imaginary and mutually conjugate.

It should be kept in mind that the first term in the second of Formulas (1.6) could have been deleted in accordance with the residue theorem, since $G_\nu(\zeta)$ decreases sufficiently rapidly for $\zeta \to \infty$. However, in order to carry out limiting transitions which are sometimes encountered it is convenient to retain this term in the general formulas.

We remark that the solutions (1.6) can be represented in finite form (without integrals) only at the wave fronts [3-5] (where their values exactly coincide with the zeroth order of approximation by the ray method), and likewise at the axis points $x = 0$ [5,7]. The investigation of the wave field for $x \neq 0$ in back of the fronts and between fronts will be our further task.

2. In the computation of the integrals (1.6) at points in back of a dilatational wave front by means of the residue theorem, it is convenient to deform the contour l into the left half-plane in such a manner that it envelopes the cuts and goes to infinity. As a result, it is not difficult to obtain the expressions

$$u_0 = u_0^\circ - \frac{2}{H}\int_m^n \operatorname{Im}\left\{G_0(i\lambda)\frac{1}{\sqrt{[\varphi_+(i\lambda)]^2+x^2}} \mp \overline{G_0(i\lambda)}\frac{1}{\sqrt{[\varphi_-(i\lambda)]^2+x^2}}\right\}d\lambda \qquad (2.1)$$

$$u_1 = u_1^\circ - \frac{2}{H}\frac{1}{x}\int_m^n \operatorname{Im}\left\{G_1(i\lambda)\left(\frac{\varphi_+(i\lambda)}{\sqrt{[\varphi_+(i\lambda)]^2+x^2}}+1\right) \mp \right.$$
$$\left. \mp \overline{G_1(i\lambda)}\left(\frac{\varphi_-(i\lambda)}{\sqrt{[\varphi_-(i\lambda)]^2+x^2}}+1\right)\right\}d\lambda \qquad (2.2)$$

in which u_ν° denotes the portion of the wave field corresponding to the residues at the possible poles of the function $G_\nu(\zeta)$ on the imaginary axis; the limits of integration m and n coincide with the smallest and largest moduli of the branch points of the functions $G_\nu(\zeta)$, with $G_\nu(i\lambda)$ being the value of $G_\nu(\zeta)$ on the right side, and $\varphi_+(i\lambda)$ and $\varphi_-(i\lambda)$ the values of $\varphi(\lambda)$ on the right and left sides respectively of the cuts emanating from the points $i\gamma_j^{-1}$.

The minus sign in the brackets corresponds to the case when $G_\nu(\zeta)$ contains a factor ζ of odd power, and the plus sign to the case when $G_\nu(\zeta)$ contains a factor ζ of even power.

In the computation of the wave field ahead of the dilatational wave front but behind the front of the head wave it is convenient to deform the contour l into the right half-plane because the cuts which in the previous case were located to the right of the imaginary axis now go over to the imaginary axis (the contour l collapses onto it). Hence this field is expressed by the integrals

$$u_0 = -\frac{4}{H}\int\limits_{\lambda'}^{n} \operatorname{Re} G_0(i\lambda)\frac{d\lambda}{\sqrt{|\varphi_+(i\lambda)|^2 - x^2}}, \qquad u_1 = \frac{4}{H}\frac{1}{x}\int\limits_{\lambda'}^{n} \operatorname{Im} G_1(i\lambda)\frac{|\varphi_+(i\lambda)|\,d\lambda}{\sqrt{|\varphi_+(i\lambda)|^2 - x^2}} \tag{2.3}$$

in which the lower limit $\lambda' > m$ is the maximum of two possible real roots of the equation

$$\varphi_+(i\lambda) - ix = 0 \tag{2.4}$$

We note that in the presence of two or more interfaces the functions $G_\nu(\zeta)$ may in some (rather infrequent) cases contain poles on the imaginary axis between branch points. In this case, if these poles fall within the interval of integration, the above integrals are to be understood in the sense of principal values.

Another form of the representation of the solution in integrals which are defined on a segment of the real axis is obtained if in the Formula (1.4) the Bessel functions are replaced by their integral representations and the integration is carried out with respect to k and ζ. This has the following form in the entire disturbed region

$$u_0 = -\frac{1}{H}\operatorname{Re}\int\limits_0^{\pi}\frac{2iG_0(\zeta_1)}{\varphi'(\zeta_1)}\,d\lambda, \qquad u_1 = -\frac{1}{H}\operatorname{Im}\int\limits_0^{\pi}\frac{2iG_1(\zeta_1)}{\varphi'(\zeta_1)}\cos\lambda\,d\lambda \tag{2.5}$$

where ζ_1 is the root of the equation

$$\varphi(\zeta) + ix\cos\lambda = 0 \tag{2.6}$$

A similar representation of the solution for the case of half-plane was first obtained in [8].

A comparison of Formulas (2.1) to (2.3) with Formulas (2.5) shows that the later (in appearance the simpler) formulas are less convenient for numerical integration. As λ varies from 0 to π, the variable ζ_1 varies over certain complex values which are different for different x, $h_j H^{-1}$, τ. In the computation of the integrals (2.5) it is in fact necessary to tabulate an extremely complicated integrand over the entire complex right half-plane. On the other hand, in the computation of the integrals (2.1) to (2.3) this tabulation is necessary only on portions of the imaginary axis. Therefore it is more convenient to use Formulas (2.1) to (2.3).

For angles of incidence less than the limiting angle, when only Formulas (2.1) to (2.2) come into play, the integrals take on finite values in the entire excited region up to the wave front. Likewise, for angles of incidence larger than the limiting angle, when Formulas (2.3) are also to be taken into account, all of the integrals take on infinite values on the dilatational wave front. This is inconvenient, both for the numerical integration in (2.1) to (2.3) (requiring a separation of the singularities), as well as in the subsequent use of the superposition principle (Duhamel integral) for the transition to an arbitrary excitation. In order to avoid these difficulties, it is convenient to tabulate the antiderivatives of the integrals with respect to τ rather than the integrals themselves. Similarly, it is then necessary to evaluate the Duhamel integrals by parts.

3. If one determines the antiderivatives with respect to τ of the integrands in Formulas (1.6), then these formulas can be written in the form

$$u_0 = \frac{1}{H}\frac{\partial}{\partial\tau}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{1}{2\zeta}\,G_0(\zeta)\ln\frac{\varphi(\zeta)+\sqrt{[\varphi(\zeta)]^2+x^2}}{\varphi(\zeta)-\sqrt{[\varphi(\zeta)]^2+x^2}}\,d\zeta \tag{3.1}$$

$$u_1 = \frac{1}{H}\frac{\partial}{\partial\tau}\frac{1}{x}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{1}{\zeta}\,G_1(\zeta)\,(\zeta\tau+\sqrt{[\varphi(\zeta)]^2+x^2})\,d\zeta \tag{3.2}$$

We remark that it turns out not to be possible to immediately determine the antiderivatives of the integrands in Formulas (2.5).

The application of the residue theorem to the integrals (3.1), (3.2) gives the following equations for the region in back of the voluminal wave front:

$$u_0 = \frac{1}{H}\frac{\partial}{\partial\tau}U_0^{\circ} + \frac{2}{H}\frac{\partial}{\partial\tau}\int_n^m\frac{1}{\lambda}\,\mathrm{Re}\left\{G_0(i\lambda)\ln\frac{\varphi_+(i\lambda)+\sqrt{[\varphi_+(i\lambda)]^2+x^2}}{ix}\mp\right.$$
$$\left.\mp\,\overline{G_0(i\lambda)}\ln\frac{\varphi_-(i\lambda)+\sqrt{[\varphi_-(i\lambda)]^2+x^2}}{ix}\right\}d\lambda \tag{3.3}$$

$$u_1 = \frac{1}{H}\frac{\partial}{\partial\tau}U_1^{\circ} + \frac{2}{H}\frac{\partial}{\partial\tau}\frac{1}{x}\int_n^m\frac{1}{\lambda}\,\mathrm{Re}\,\{G_1(i\lambda)\,[i\lambda\tau+\sqrt{[\varphi_+(i\lambda)^2+x^2}]\mp$$
$$\mp\,\overline{G_1(i\lambda)}\,[i\lambda\tau+\sqrt{[\varphi_-(i\lambda)]^2+x^2}\}\,d\lambda \tag{3.4}$$

Here $\partial U_\nu{}^{\circ}/\partial\tau$ ($\nu = 0, 1$) denote terms corresponding to residues of the possible poles of the functions $G_\nu(\zeta)\,\zeta^{-1}$.

In Formulas (3.3), (3.4) the upper or lower signs are to be chosen by the same rule as in Formulas (2.1), (2.2).

For the region between the voluminal and head wave fronts we have

$$u_0 = \frac{4}{H}\frac{\partial}{\partial\tau}\int_{\lambda'}^{n}\frac{1}{\lambda}\,\mathrm{Re}\left[G_0(i\lambda)\ln\frac{|\varphi_+(i\lambda)| - \sqrt{|\varphi_+(i\lambda)|^2 - x^2}}{x}\right]d\lambda \qquad (3.5)$$

$$u_1 = -\frac{4}{H}\frac{\partial}{\partial\tau}\frac{1}{x}\int_{\lambda'}^{n}\frac{1}{\lambda}\,\mathrm{Im}\,[G_1(i\lambda)\sqrt{|\varphi_+(i\lambda)|^2 - x^2}]\,d\lambda \qquad (3.6)$$

where λ' is the same as in Formulas (2.3).

4. We give now the simplest application of some of the formulas examined above. We investigate the wave picture on the boundary $z = h$ separating two media. For the sake of simplifying the computation we shall assume that the density ρ and the wave velocity of transverse waves b (and hence the shear modulus as well) are equal in the two media. However, the propagation velocities of the longitudinal waves a_0 and a_1 are different. We place a source in the form of a center of expansion at the origin of coordinates in the medium $z < h$ with the larger propagation velocity $a_0 > a_1$.

For the given conditions the displacement field at the boundary $z = h$ may be represented as the sum of fields of incident and reflected longitudinal waves. The well known field of the incident waves, which is expressible in finite form, is not characterized by any singularities in back of the wave front. Hence, in the sequel we are concerned only with the field of vertical and horizontal displacements, w and q, in the reflected wave. For an excitation whose time variation is that of the Heaviside ε function, it can be represented in accordance with [3] in the form of the following integrals

$$w = \frac{\gamma^2}{8\pi^2\mu h^2}\frac{\partial}{\partial\tau}\int_0^\infty\left\{\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\alpha_1-\alpha_0}{i(\alpha_1+\alpha_0)\zeta^2}\,e^{k(\tau\zeta-\alpha_0)}\,d\zeta\right\}J_0(kx)\,dk \qquad (4.1)$$

$$q = \frac{-\gamma^2}{8\pi^2\mu h^2}\frac{\partial}{\partial\tau}\frac{1}{x}\int_0^\infty\left\{\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\alpha_1-\alpha_0}{i(\alpha_1+\alpha_0)\alpha_0\zeta^2}\,e^{k(\tau\zeta-\alpha_0)}\,d\zeta\right\}J_1(kx)\,dk \qquad (4.2)$$

Here

$$\alpha_0 = \sqrt{1+\gamma^2\zeta^2}, \quad \alpha_1 = \sqrt{1+\Delta^2\zeta^2}, \quad \gamma = \frac{b}{a_0}, \quad \Delta = \frac{b}{a_1}, \quad x = \frac{r}{h}, \quad \tau = \frac{bt}{h} \qquad (4.3)$$

If Formulas (2.2) and (2.2) are applied and it is taken into account that the integrands in (4.1), (4.2) do not have poles, then one obtains the equations

$$w = \frac{\gamma^2}{\pi^2 \mu h^2 (\Delta^2 - \gamma^2)} \frac{\partial}{\partial \tau} \int_{1/\Delta}^{1/\gamma} \frac{|\alpha_0||\alpha_1|}{\lambda^4} \operatorname{Im} \frac{1}{\sqrt{(i\lambda\tau - |\alpha_0|)^2 + x^2}} \, d\lambda \tag{4.4}$$

$$q = \frac{-\gamma^2}{\pi^2 \mu h^2 (\Delta^2 - \gamma^2)} \frac{\partial}{\partial \tau} \frac{1}{x} \int_{1/\Delta}^{1/\gamma} \frac{|\alpha_1|}{\lambda^4} \operatorname{Im} \frac{i\lambda\tau - |\alpha_0|}{\sqrt{(i\lambda\tau - |\alpha_0|)^2 + x^2}} \, d\lambda \tag{4.5}$$

in which

$$|\alpha_0| = \sqrt{1 - \gamma^2\lambda^2}, \qquad |\alpha_1| = \sqrt{\Delta^2\lambda^2 - 1} \tag{4.6}$$

and the argument of the radical $\sqrt{(i\lambda\tau - |\alpha_0|^2 + x^2)}$ should be in the interval $(0, -\pi/2)$.

At the wave front (at $\tau = \gamma\sqrt{(1 + x^2)}$ the integrals in Formulas (4.4), (4.5) are of finite form. Performing these integrations, we find the following values for the antiderivatives of the displacements

$$\int^{\aleph} w(\tau)\, d\tau = \frac{\gamma^3}{4\pi\mu h^2 (\Delta^2 - \gamma^2)} \frac{\Delta^2 + \gamma^2 + (\Delta^2 - \gamma^2)x^2 - 2\gamma\sqrt{\Delta^2 + (\Delta^2 - \gamma^2)x^2}}{(1 + x^2)^2} \tag{4.7}$$

$$\int^{\aleph} q(\tau)\, d\tau = -x \int^{\aleph} w(\tau)\, d\tau \qquad (\aleph = \gamma\sqrt{1 + x^2} + 0) \tag{4.8}$$

Formula (4.8) shows that for large values of x the horizontal components of the displacements at the wave front will be an order of magnitude larger than the corresponding vertical displacements.

Calculations were carried out according to Formulas (4.4), (4.5), (4.7) and (4.8) for the values $\gamma = 1/3$, $\Delta = 1/2$, $x = 10$. The trapezoidal quadrature formula was used for the interval of integration divided into ten parts.

We give here the results of the calculations of the antiderivatives (the factor $8/10^5 \mu(\pi h)^2$ is omitted) w^* and q^* of the field of displacements w and q for certain values of the argument $\tau = bt/h$:

$\tau =$	3,35	4	4.5	5	5.5	6	8	10
$w^* =$	3,05	11.3	14.5	13.1	10.2	8.48	5.32	3.98
$q^* =$	30,5	21.4	12.4	4.40	1.71	1.01	0.210	0.0840

The computations show that instants of time close to $\tau = 5$ are characterized by a sharp change in the wave field, corresponding to a head wave of surface type [9]. The so-called transverse "nondiscontinuous" surface waves discussed in [10-14] are related to this kind of wave.

Earlier examples wherein exact solutions are transformed into real integrals which are evaluated by quadrature formulas may be found in [5, 13, 15-17] (for particular, simpler problems). In contrast to previous results, the formulas derived here are characterized by great generality and rid one of the necessity of separating out the singularities on the wave fronts.

In the literature there are many efforts devoted to various approximate asymptotic investigations (often without regard of the errors involved) of reflected, refracted, and head waves for sharp and weak interfaces; to estimates of oscillations propagating at angles which are close to the limiting angle; to the study of shielding; to the determination of the wave field in the neighborhood of a source, etc. All of these problems, as well as a number of others (particularly when it is difficult to apply a reliable asymptotic method), can be solved to an arbitrary degree of accuracy by means of a single numerical scheme that utilizes the formulas that have been introduced. Further, a significant part of the numerical work done for one problem can often be used in the solution of other problems.

BIBLIOGRAPHY

1. Smirnov, V.I. and Sobolev, S.L., O primenenii novogo metoda k izucheniiu uprugikh voln v prostranstve pri nalichii osevoi simmetrii (On the application of a new method to the study of elastic waves in three dimensions in the presence of axial symmetry). *Tr. Seismologich. in-ta Akad. Nauk SSSR* No. 29, 1933.

2. Cagniard, L., *Reflexion et refractions des onds seismiques progressives.* Gauthier-Villars, Paris, 1933.

3. Petrashen', G.I., Metodika postroeniia reshenii zadach na rasprostranenie seismicheskikh voln v izotropnykh sredakh, soderzhashchikh tolstye ploskoparallel'nye sloi (spravochnik) (Methodology of constructing solutions of problems on the propagation of seismic waves in isotropic media containing thick planar parallel layers (reference book)). *Vopr. dinamicheskoi teorii rasprostraneniia seismicheskikh voln* (Problems *in the Dynamic Theory of the Propagation of Seismic Waves*). Sb. 1, 1957.

4. Zvolinskii, N.V., Otrazhennye i golovnye volny, voznikaiushchie na ploskoi granitse razdela dvukh uprugikh sred (Reflected and head waves excited at the plane boundary separating two elastic media), I, II. *Izv. Akad. Nauk SSSR, ser. geofiz.* No.10, 1957; No.1, 1958.

5. Ogurtsov, K.I. and Petrashen', G.I., Dinamicheskie zadachi dlia uprugogo poluprostranstva v sluchae osevoi simmetrii (Dynamic problems for an elastic half-space in the case of axial symmetry). *Uch. zap. LGU* No. 149, 1951.

6. Grey, E. and Matthews, G.B., *Funktsii Besselia i ikh prilozheniia k fizike i mekhanike (Bessel Functions and their Applications to Physics and Mathematics)*. IIL, 1953.

7. Ogurtsov, K.I., Otsenki intensivnostei seismicheskikh voln, otrazivshikhsia ot ochen' slabykh granits razdela (Estimates of the magnitude of seismic waves reflected from very weak separation boundaries). *Izv. Akad. Nauk SSSR, ser. geofiz.* No. 10, 1960.

8. Sobolev, S.L., O primenenii teorii ploskikh voln k zadache Lemba (On the application of the theory of plane waves to the Lamb problem). *Tr. Seismologich. In-ta Akad. Nauk SSSR* No. 18, 1932.

9. Zaitsev, L.P., O golovnoi volne poverkhnostnogo tipa (On a head wave of surface wave type). *Vopr. dinamicheskoi teorii rasprostraneniia seismicheskikh voln (Problems in the Dynamic Theory of the Propagation of Seismic Waves)*. Sb. III, 1959.

10. Jeffrey, H., Seismology. *Rep. Prog. Phys.* 10, 52, 1946.

11. Lapwood, E.R., The disturbance due to a line source in a semi-infinite elastic medium. *Phil. Trans.* A242, 63, 1949.

12. Newlands, M., The disturbance due to a line source in a semi-infinite elastic medium with a single surface layer. *Phil. Trans.* A245, 213, 1952.

13. Pinney, E., Surface motion due to a point source in a semi-infinite elastic medium. *Bulletin of the Seismological Society of America* Vol. 44, No. 4, 571-596, 1954.

14. Ogurtsov, K.I., Utochnenie kharaktera kolebanii tochek granitsy uprugogo poluprostranstva dlia dvukh naibolee raprostranennykh vozdeistvii (A refinement of the character of the oscillations of points on the boundary of an elastic half-space for the two fastest propagating disturbances). *Vopr. rudnoi geofiziki* No. 1. Gosgeotekhizdat, 1960.

15. Ogurtsov, K.I., Kolichestvennye issledovaniia volnovkh protsessov v uprugom poluprostranstve pri razlichnykh tipakh vozdeistvii (Quantitative investigations of wave processes in an elastic half-space for various types of disturbances). *Uch. zap. LGU* No. 208, 1956.

16. Pekeris, S.L., The seismic surface pulse. *Proc. Nat. Acad. Sci. USA* Vol. 41, No. 7, 469-480, 1955.

17. Shemiakin, E.I., Zadacha Lemba dlia vnutrennego istochnika (Lamb's problem for an internal source). *Dokl. Akad. Nauk SSSR* Vol. 140, No. 4, 780-782, 1961.

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